Phys 410
Spring 2013
Lecture #23 Summary
19 November, 2013

We considered another coupled oscillator problem – the double pendulum. We wrote down the Lagrangian, which turned out to be quite complicated. It leads to nonlinear equations of motion – as is well known for the single pendulum. To avoid this problem (which we will deal with later), we made a "small oscillations" approximation for the double pendulum. In this approximation we take ϕ_1, ϕ_2, $\dot{\phi}_1$, and $\dot{\phi}_2$ to be small, and only keep terms up to second order in these quantities. We then did a Taylor series expansion for the kinetic energy and potential energy to arrive at an approximate Lagrangian of the form: $\mathcal{L} = \frac{1}{2}(m_1 + m_2)\left(L_1\dot{\phi}_1\right)^2 + m_2 L_1 L_2 \dot{\phi}_1\dot{\phi}_2 + \frac{1}{2}m_2\left(L_2\dot{\phi}_2\right)^2 - \frac{(m_1+m_2)gL_1\phi_1^2}{2} - \frac{m_2 g L_2 \phi_2^2}{2}$. Both the kinetic energy and the potential energy are homogeneous quadratic functions.

We then used Lagrange's equations to find the equations of motion for the two generalized coordinates ϕ_1, ϕ_2, with the following results:

ϕ_1-equation: $-(m_1 + m_2)gL_1\phi_1 = (m_1 + m_2)L_1^2\ddot{\phi}_1 + m_2 L_1 L_2 \ddot{\phi}_2$

ϕ_2-equation: $-m_2 g L_2 \phi_2 = m_2 L_1 L_2 \ddot{\phi}_1 + m_2 L_2^2 \ddot{\phi}_2$

These two equations can be summarized in matrix form as $\overline{\overline{M}}\ddot{\vec{\phi}} = -\overline{\overline{K}}\vec{\phi}$, with $\vec{\phi} = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$, $\overline{\overline{M}} = \begin{pmatrix}(m_1 + m_2)L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2\end{pmatrix}$ and $\overline{\overline{K}} = \begin{pmatrix}(m_1 + m_2)gL_1 & 0 \\ 0 & m_2 g L_2\end{pmatrix}$. The "mass matrix" is now made up of rotational inertia terms, while the "spring constant matrix" is made up of restoring torque terms. Note that the K-matrix is diagonal, whereas the M-matrix is not – this is the opposite of the situation for the 2-mass-3-spring problem, showing that we have a different kind of coupling here. We again use the complex ansatz for the solution vector: $\vec{\phi}(t) = Re\left[\vec{C}e^{i\omega t}\right]$, where $\vec{C} = \begin{pmatrix}C_1 \\ C_2\end{pmatrix}$, and C_1 and C_2 are complex constants. Putting this into the matrix equation yields $\left(\overline{\overline{K}} - \omega^2\overline{\overline{M}}\right)\vec{C} = 0$. To get a non-trivial solution for $\vec{C}$, we demand that $det\left(\overline{\overline{K}} - \omega^2\overline{\overline{M}}\right) = 0$. This yields a quadratic equation for ω^2, with two solutions.

We then considered the special case of a double pendulum with equal masses (m) and equal lengths (L), and introduce the natural frequency ($\omega_0^2 \equiv g/L$). The determinant yields two normal mode frequency solutions: $\omega_1 = \omega_0\sqrt{2 - \sqrt{2}}$, and $\omega_2 = \omega_0\sqrt{2 + \sqrt{2}}$. The corresponding normal modes are the analogs of the "sloshing" and "beating" modes. The

first is of the form $\vec{\phi} = A_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \cos(\omega_1 t - \delta_1)$, while the second is $\vec{\phi} = A_2 \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \cos(\omega_2 t - \delta_2)$. In the first normal mode the two pendula swing together in phase (the sloshing mode), with the lower pendulum swinging with greater amplitude. In the other mode the two pendula swing 180^o out of phase (a type of beating mode).

We then went on to consider the most general coupled oscillator problem – N particles coupled to each other by means of springs or any other types of forces that produce a stable equilibrium configuration. This system has n generalized coordinates, where in general $n \neq N$. The generalized coordinates are written as $\vec{q} = (q_1, q_2, \ldots q_n)$. We assume that only conservative forces act between the particles, hence (as known from previous studies) the potential energy is a function only of the coordinates: $U = U(\vec{q})$. The kinetic energy is that of all of the particles in the system: $T = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha^2$. The "raw" coordinates $\vec{r}_\alpha$can be written in terms of the generalized coordinates as $\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2, \ldots q_n)$, where it is assumed that no explicit time-dependence is required to write down this transformation. The kinetic energy can be written as $T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\, \dot{q}_i \dot{q}_j$, where the matrix $\bar{\bar{A}}$ is defined as $A_{ij} \equiv \sum_{\alpha=1}^{N} m_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i}\frac{\partial \vec{r}_\alpha}{\partial q_j}$. Note that the double pendulum kinetic energy (see the Lagrangian above) has a kinetic energy of this form, including a $\dot{q}_1\dot{q}_2$ term. Note that the matrix $\bar{\bar{A}}$ is a function of the generalized coordinates as well: $\bar{\bar{A}} = \bar{\bar{A}}(\vec{q})$. We now have the full Lagrangian of this generalized coupled oscillator problem $\mathcal{L} = T(\vec{q}, \dot{\vec{q}}) - U(\vec{q})$.

We next considered the small oscillation motion of the system around a stable equilibrium point. This means that we will keep terms only up to second order in the variables. By a shift of the origin, we can make the stable equilibrium point appear at the point $\vec{q} = (0, 0, \ldots 0)$. We then did a Taylor series expansion of the potential around this point and kept terms up to second order, yielding $U(\vec{q}) = \frac{1}{2}\sum_{i,j} K_{ij} q_i q_j$, where the matrix elements of $\bar{\bar{K}}$ are the curvatures of the potential with respect to the generalized coordinates: $K_{ij} \equiv \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{\vec{q}=0}$. The kinetic energy is already quadratic in the variables, so we simply evaluate it at $\vec{q} = 0$ to yield $T = \frac{1}{2}\sum_{i,j} A_{ij}(0)\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i,j} M_{ij}\dot{q}_i\dot{q}_j$, where the mass matrix $\bar{\bar{M}}$ is the $\bar{\bar{A}}$ matrix evaluated at the equilibrium position $\vec{q} = (0, 0, \ldots 0)$. The Lagrangian $\mathcal{L} = T(\dot{\vec{q}}) - U(\vec{q})$ is now a homogeneous quadratic function of the coordinates and their time-derivatives, and the matrices $\bar{\bar{M}}$ and $\bar{\bar{K}}$ are constant symmetric real matrices.

There are n Lagrange equations to set up and solve. We wrote down the equations and found that the set of n equations are summarized beautifully in a simple matrix equation: $-\bar{\bar{K}}\vec{q} = \bar{\bar{M}}\ddot{\vec{q}}$. We can solve this equation using the same method employed before, just

generalized to n coordinates. We use the standard complex *ansatz* for the solution vector: $\vec{q}(t) = Re\left[\vec{C}e^{i\omega t}\right]$, where $\vec{C} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}$, and the C_i are complex constants. Putting this into the matrix equation yields $\left(\overline{\overline{K}} - \omega^2\overline{\overline{M}}\right)\vec{C} = 0$. To get a non-trivial solution for $\vec{C}$, we demand that $det\left(\overline{\overline{K}} - \omega^2\overline{\overline{M}}\right) = 0$. This yields an n-th order equation for ω^2, with n real solutions (we know this because the matrix $\overline{\overline{K}} - \omega^2\overline{\overline{M}}$ is real and symmetric). The n normal modes follow by standard linear algebra. The most general solution is a linear combination of motion in all of the normal modes, each with distinct amplitude and phase. The motion in a given normal mode may involve a coordinated motion of all the particles in the system!